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Properties of the Ray Transforms of Two-Dimensional 2-Tensor Fields Defined in the Unit Disk

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Abstract—We study the longitudinal, transverse, and mixed ray transforms acting on two-dimensional symmetric 2-tensor fields. Namely, the kernels of the ray transforms are described; the connection between the ray transforms and the Radon transform is established; some unconditional estimates of stability for each of the ray transforms are obtained; inversion reconstruction formulas for the components of the symmetric 2-tensor and for the recovery of the potential are deduced; and the projection theorems for the ray transforms are proved as well.

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In inverse problems, the sought quantities are often not scalar functions but vector or tensor fields of different valences. Such are the statements of a number of problems in the theory of nonhomogeneous and anisotropic media, gas dynamics, hydrodynamics, and electrodynamics. The mathematical statements of the problems of reconstructing vector and tensor fields appeared relatively recently (for example, see [1]). Their further development led to the formulation of inverse problems with tomographic data type, which are natural to consider as applications of the integral geometry of scalar [2] and tensor fields on a Riemannian manifold [3]. The properties of the Radon transform (the ray transform of scalar fields) are well known and are described, for example, in [4]. At the same time, the properties of ray transforms acting on tensor fields are not completely studied.

In [3], the properties were investigated of the longitudinal ray transform acting on symmetric tensor fields of an arbitrary valence m in media with refraction. It was proved in particular that the kernel consists of the potential fields with potentials vanishing on the boundary, and the estimates of stability were given which are conditional for $m \geq 1$. The componentwise inversion formulas were obtained in the case of the Euclidean metric.

Note the article [5] which provides a study of the properties of the longitudinal and transverse ray transforms acting on two-dimensional vector fields. It was proved in [5] that the kernel of the transverse ray transform consists of the solenoidal vector fields with potentials vanishing on the boundary. In the case of the Euclidean metric, the connection between the longitudinal and transverse ray transforms and the Radon transform was pointed out, unconditional stability estimates for both ray transforms were obtained, inversion formulas for the reconstruction of the components of the vector field and the reconstruction of the potential were written.

In practice, all fields studied by the tomographic methods belong to a bounded domain. Namely, it is assumed that the field is identical zero outside the domain. In the present article, we investigate the properties of the operators of the longitudinal, transverse, and symmetric ray transforms acting on the two-dimensional symmetric 2-tensor fields defined on the unit circle in the case of the Euclidean metric.

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1. DEFINITIONS

Consider the cylinder $Z = [-1,1] \times [0,2\pi]$ and the unit disk $B = \{x \in \mathbb{R}^2 \mid (x^1)^2 + (x^2)^2 < 1\}$ with the boundary $\partial B = \{x \in \mathbb{R}^2 \mid (x^1)^2 + (x^2)^2 = 1\}$.

1.1. The Spaces Used

Let $f(x), g(x), \ldots$ denote the functions (scalar fields) and let $\varphi(x), \psi(x), \phi(x), \ldots$ designate the potential defining vector and 2-tensor fields. The set of symmetric m-tensor fields $w(x) = (w_{i_1\ldots i_m}(x))$, $u(x) = (u_{i_1\ldots i_m}(x))$, $v(x) = (v_{i_1\ldots i_m}(x)), \ldots$, where $i_1, \ldots, i_m = 1, 2$, defined in B is denoted by $S^m(B)$ (in this article, $m = 0, 1, 2$). The scalar product in $S^m(B)$ is introduced by the formula

$$\langle u(x), v(x)\rangle = u_{i_1\ldots i_m}(x) v^{i_1\ldots i_m}(x).$$

Henceforth, repeated super- and subscripts in a monomial imply summation from 1 to 2. Recall that, in a Euclidean space with a Cartesian rectangular coordinate system, there is no difference between contravariant and covariant components.

We need the spaces of square integrable functions $L_2(B)$ and symmetric m-tensor fields $L_2(S^m(B))$ as well as the $L_2(Z)$ space. The inner product on $L_2(S^m(B))$ is defined as

$$(u, v)_{L_2(S^m(B))} = \int\limits_B \langle u(x), v(x)\rangle\, dx.$$

The spaces of differentiable symmetric m-tensor fields with finite order k are designated by $C^k(S^m(B))$ and $C_0^k(S^m(B))$; the Sobolev spaces are denoted by $H^k(S^m(B))$, $H_0^k(S^m(B))$, and $H^k(Z)$. Denote the space of infinitely differentiable functions by $C^\infty(B)$.

1.2. Differential Operators

The operators of inner differentiation d and inner $\perp$-differentiation $d^\perp$ are the compositions of operators of covariant derivation and symmetrization

$$d, d^\perp : H^k(S^m(B)) \to H^{k-1}(S^{m+1}(B))$$

and act on a function f and a vector field v by the rules

$$(df)_i = \frac{\partial f}{\partial x^i}, \qquad (dv)_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x^j} + \frac{\partial v_j}{\partial x^i}\right),$$
$$(d^\perp f)_i = (-1)^i \frac{\partial f}{\partial x^{3-i}}, \qquad (d^\perp v)_{ij} = \frac{1}{2}\left((-1)^j \frac{\partial v_i}{\partial x^{3-j}} + (-1)^i \frac{\partial v_j}{\partial x^{3-i}}\right).$$

The divergence operator $\delta : H^k(S^m(B)) \to H^{k-1}(S^{m-1}(B))$ acts on a vector field v and on a symmetric 2-tensor field w by the formulas

$$\delta v = \frac{\partial v_j}{\partial x^j}, \qquad (\delta w)_i = \frac{\partial w_{ij}}{\partial x^j}. \tag{1}$$

Recall that an m-tensor field $u \in H^k(S^m(B))$ is called *potential* if there exists an $(m-1)$-tensor field $v \in H^{k+1}(S^{m-1}(B))$ (a *potential*) such that $u = dv$. A field $w \in H^k(S^m(B))$ is called *solenoidal* if $\delta w = 0 \in H^{k-1}(S^{m-1}(B))$. Every two-dimensional solenoidal symmetric tensor field can be represented by means of one function [6]. In particular, for every solenoidal two-dimensional vector field $v \in H^k(S^1(B))$, there exists a potential $\psi \in H^{k+1}(B)$ such that $d^\perp\psi = v$; and every solenoidal two-dimensional 2-tensor field $u \in H^k(S^2(B))$ can be represented as $u = (d^\perp)^2\psi$ for some $\psi \in H^{k+2}(B)$ (which is checked by insertion into (1)).

It is known [3] that, for every $w \in H^k(S^m(B))$, there exists a solenoidal tensor field $v \in H^k(S^m(B))$ and a potential $u \in H^{k+1}(S^{m-1}(B))$ such that

$$w = v + du, \qquad u|_{\partial B} = 0, \qquad \delta v = 0. \tag{2}$$

This decomposition is unique.

Using (2) for $m = 1$ and $m = 2$, the theorem on the decomposition of a vector field [7], and the representation of a solenoidal vector field in terms of the potential, we obtain one of the versions of a decomposition of a symmetric 2-tensor field. We have the unique decomposition for every 2-tensor field $u \in L_2(S^2(B))$:

$$u = d^2\varphi + dd^\perp\phi + (d^\perp)^2\psi, \qquad \varphi, \phi, \psi \in H^2(B), \tag{3}$$

$$\varphi|_{\partial B} = \frac{\partial\varphi}{\partial x^1}\bigg|_{\partial B} = \frac{\partial\varphi}{\partial x^2}\bigg|_{\partial B} = 0, \qquad \frac{\partial\phi}{\partial x^1}\bigg|_{\partial B} = \frac{\partial\phi}{\partial x^2}\bigg|_{\partial B} = 0. \tag{4}$$

In the present article, we consider only the fields for which the potentials φ, ϕ, and ψ in (3) and (4) vanish on ∂B together with their first derivatives; i.e., $\varphi, \phi, \psi \in H_0^2(B)$.

1.3. The Radon Transform and the Ray Transforms

The *Radon transform* of a function f is the operator $\mathcal{R} : H^k(B) \to H^k(Z)$ defined as

$$(\mathcal{R}f)(s,\theta) = \int_{-\infty}^{\infty} f(t\xi + s\eta)\,dt = \int_{-\sqrt{1-s^2}}^{\sqrt{1-s^2}} f(t\xi + s\eta)\,dt.$$

Here $\xi = (-\sin\theta, \cos\theta)$ is the direction vector of the straight line along which integration is carried out; and $\eta = (\xi^2, -\xi^1) = (\cos\theta, \sin\theta)$ is the normal vector.

The *longitudinal* $\mathcal{P}$, *transverse* $\mathcal{P}^\perp$, and *mixed* $\mathcal{P}^\star$ *ray transforms*

$$\mathcal{P}, \mathcal{P}^\perp, \mathcal{P}^\star : H^k(S^2(B)) \to H^k(Z)$$

of a symmetric 2-tensor field w are defined as

$$(\mathcal{P}w)(s,\theta) = \int_{-\infty}^{\infty} w_{ij}(t\xi + s\eta)\xi^i\xi^j\,dt = \int_{-\sqrt{1-s^2}}^{\sqrt{1-s^2}} w_{ij}(t\xi + s\eta)\xi^i\xi^j\,dt,$$

$$(\mathcal{P}^\perp w)(s,\theta) = \int_{-\infty}^{\infty} w_{ij}(t\xi + s\eta)\eta^i\eta^j\,dt = \int_{-\sqrt{1-s^2}}^{\sqrt{1-s^2}} w_{ij}(t\xi + s\eta)\eta^i\eta^j\,dt,$$

$$(\mathcal{P}^\star w)(s,\theta) = \int_{-\infty}^{\infty} w_{ij}(t\xi + s\eta)\eta^i\xi^j\,dt = \int_{-\sqrt{1-s^2}}^{\sqrt{1-s^2}} w_{ij}(t\xi + s\eta)\eta^i\xi^j\,dt.$$

2. PROPERTIES OF THE RAY TRANSFORMS

The kernels of the ray transforms of 2-tensor fields satisfy

Theorem 1. *The following hold for every function $\varphi \in H_0^2(B)$:*

$$\mathcal{P}(d^2\varphi) = \mathcal{P}(dd^\perp\varphi) = 0, \qquad \mathcal{P}^\perp((d^\perp)^2\varphi) = \mathcal{P}^\perp(dd^\perp\varphi) = 0,$$
$$\mathcal{P}^\star((d^\perp)^2\varphi) = \mathcal{P}^\star(d^2\varphi) = 0.$$

Proof. The equalities $\mathcal{P}(d^2\varphi) = \mathcal{P}(dd^\perp\varphi) = 0$ are well known (for instance, see [3]). Let us prove only that $\mathcal{P}^\perp((d^\perp)^2\varphi) = 0$ since the remaining equalities are established similarly.

By the definition of $\mathcal{P}^\perp$, we have

$$(\mathcal{P}^\perp((d^\perp)^2\varphi))(s,\theta) = \int\limits_{-\sqrt{1-s^2}}^{\sqrt{1-s^2}} \left(\frac{\partial^2\varphi}{\partial(x^2)^2}\eta^1\eta^1 - \frac{\partial^2\varphi}{\partial x^1\partial x^2}\eta^1\eta^2 - \frac{\partial^2\varphi}{\partial x^1\partial x^2}\eta^1\eta^2 + \frac{\partial^2\varphi}{\partial(x^1)^2}\eta^2\eta^2 \right) dt. \quad (5)$$

Note that

$$\eta^1 = \frac{\partial x^2}{\partial t}, \qquad \eta^2 = -\frac{\partial x^1}{\partial t}, \qquad \frac{\partial\eta^1}{\partial t} = \frac{\partial\eta^2}{\partial t} = 0.$$

Inserting these into (5), we infer

$$\begin{aligned}(\mathcal{P}^\perp((d^\perp)^2\varphi))(s,\theta) &= \int\limits_{-\sqrt{1-s^2}}^{\sqrt{1-s^2}} \left(\frac{\partial^2\varphi}{\partial(x^2)^2}\frac{\partial x^2}{\partial t}\eta^1 - \frac{\partial^2\varphi}{\partial x^1\partial x^2}\frac{\partial x^2}{\partial t}\eta^2 + \frac{\partial^2\varphi}{\partial x^1\partial x^2}\frac{\partial x^1}{\partial t}\eta^1 - \frac{\partial^2\varphi}{\partial(x^1)^2}\frac{\partial x^1}{\partial t}\eta^2 \right) dt \\ &= \int\limits_{-\sqrt{1-s^2}}^{\sqrt{1-s^2}} \frac{d}{dt}\left(\frac{\partial\varphi}{\partial x^2}\eta^1 - \frac{\partial\varphi}{\partial x^1}\eta^2 \right) dt = \left(\frac{\partial\varphi}{\partial x^2}\eta^1 - \frac{\partial\varphi}{\partial x^1}\eta^2 \right)\Bigg|_{-\sqrt{1-s^2}}^{\sqrt{1-s^2}} = 0,\end{aligned}$$

which completes the proof. □

The following describes the relationship between the ray transforms and the Radon transform:

Theorem 2. *For every $\varphi \in H_0^2(B)$, we have:*

$$\mathcal{P}((d^\perp)^2\varphi) = \mathcal{P}^\perp(d^2\varphi) = 2\mathcal{P}^\star(dd^\perp\varphi) = \frac{\partial^2}{\partial s^2}(\mathcal{R}\varphi).$$

Proof. Since

$$\varphi|_{\partial B} = \frac{\partial\varphi}{\partial x^1}\bigg|_{\partial B} = \frac{\partial\varphi}{\partial x^2}\bigg|_{\partial B} = 0,$$

we have

$$\begin{aligned}\frac{\partial^2}{\partial s^2}(\mathcal{R}\varphi)(s,\theta) &= \int\limits_{-\sqrt{1-s^2}}^{\sqrt{1-s^2}} \left[\frac{\partial^2\varphi}{\partial(x^1)^2}\cos^2\theta + 2\left(\frac{\partial^2\varphi}{\partial x^1\partial x^2}\right)\sin\theta\cos\theta + \frac{\partial^2\varphi}{\partial(x^2)^2}\sin^2\theta \right] dt \\ &= \int\limits_{-\sqrt{1-s^2}}^{\sqrt{1-s^2}} \left[\frac{\partial^2\varphi}{\partial(x^1)^2}(\xi^2)^2 + 2\left(\frac{\partial^2\varphi}{\partial x^1\partial x^2}\right)(-\xi^1)\xi^2 + \frac{\partial^2\varphi}{\partial(x^2)^2}(-\xi^1)^2 \right] dt = \big(\mathcal{P}((d^\perp)^2\varphi)\big)(s,\theta).\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{\partial^2}{\partial s^2}(\mathcal{R}\varphi)(s,\theta) &= \int\limits_{-\sqrt{1-s^2}}^{\sqrt{1-s^2}} \left[\frac{\partial^2\varphi}{\partial(x^1)^2}(\eta^1)^2 + 2\left(\frac{\partial^2\varphi}{\partial x^1\partial x^2}\right)\eta^1\eta^2 + \frac{\partial^2\varphi}{\partial(x^2)^2}(\eta^2)^2 \right] dt \\ &= \big(\mathcal{P}^\perp(d^2\varphi)\big)(s,\theta).\end{aligned}$$

Now, consider $(\mathcal{P}^\star(dd^\perp\varphi))$:

$$(\mathcal{P}^\star(dd^\perp\varphi))(s,\theta) = \int\limits_{-\sqrt{1-s^2}}^{\sqrt{1-s^2}} \left[2\frac{\partial^2\varphi}{\partial x^1\partial x^2}\sin\theta\cos\theta + \frac{1}{2}\left(\frac{\partial^2\varphi}{\partial(x^1)^2} - \frac{\partial^2\varphi}{\partial(x^2)^2}\right)\left(\cos^2\theta - \sin^2\theta\right)\right] dt.$$

Hence,

$$(\mathcal{P}((d^\perp)^2\varphi))(s,\theta) - (\mathcal{P}^\star(dd^\perp\varphi))(s,\theta) = \frac{1}{2}\int\limits_{-\sqrt{1-s^2}}^{\sqrt{1-s^2}} \left(\frac{\partial^2\varphi}{\partial(x^1)^2} + \frac{\partial^2\varphi}{\partial(x^2)^2}\right) dt = \frac{1}{2}(\mathcal{R}(\Delta\varphi))(s,\theta).$$

As is well known [4],

$$(\mathcal{R}(\Delta\varphi))(s,\theta) = \frac{\partial^2}{\partial s^2}(\mathcal{R}\varphi)(s,\theta);$$

therefore,

$$(\mathcal{P}^\star(dd^\perp\varphi))(s,\theta) = \frac{1}{2}\frac{\partial^2}{\partial s^2}(\mathcal{R}\varphi)(s,\theta).$$

This completes the proof. □

Suppose that $f \in L_2(B)$ is a function and $u \in H^1(S^2(B))$ is a symmetric 2-tensor field $u = (d^\perp)^2\psi + dd^\perp\phi + d^2\varphi$ with potentials $\varphi,\phi,\psi \in H_0^2(B)$. In [3], there were proved the stability estimates

$$\|f\|^2_{L_2(B)} \le C_1\|\mathcal{R}f\|^2_{H^1(Z)}, \tag{6}$$

$$\|(d^\perp)^2\psi\|^2_{L_2(S^2(B))} \le C_2\big(\|u\|_{H^1(S^2(B))}\|\mathcal{P}u\|_{L_2(Z)} + \|\mathcal{P}u\|^2_{H^1(Z)}\big) \tag{7}$$

with the constants C_1 and C_2 independent of f and u respectively.

Decomposition (3), (4), estimate (6), and the properties of the ray transforms give a stability estimate stronger than (7) for the operator $\mathcal{P}$ and also estimates for $\mathcal{P}^\perp$ and $\mathcal{P}^\star$:

Theorem 3. *Let $u = (d^\perp)^2\psi + dd^\perp\phi + d^2\varphi$ with potentials $\varphi,\phi,\psi \in H_0^2(B)$. Then we have the stability estimates*

$$\|(d^\perp)^2\psi\|^2_{L_2(S^2(B))} \le C\|\mathcal{P}u\|^2_{H^1(Z)}, \tag{8}$$

$$\|d^2\varphi\|^2_{L_2(S^2(B))} \le C^\perp\|\mathcal{P}^\perp u\|^2_{H^1(Z)}, \tag{9}$$

$$\|dd^\perp\phi\|^2_{L_2(S^2(B))} \le C^\star\|\mathcal{P}^\star u\|^2_{H^1(Z)}, \tag{10}$$

where C, $C^\perp$, and $C^\star$ are constants independent of u.

Proof. Starting from the definitions of $\mathcal{P}$, $\mathcal{P}^\perp$, and $\mathcal{P}^\star$ for an arbitrary symmetric 2-tensor field v, we have

$$(\mathcal{P}v) = (\mathcal{R}v_{11})\sin^2\theta - 2(\mathcal{R}v_{12})\sin\theta\cos\theta + (\mathcal{R}v_{22})\cos^2\theta,$$

$$(\mathcal{P}^\perp v) = (\mathcal{R}v_{11})\cos^2\theta + 2(\mathcal{R}v_{12})\sin\theta\cos\theta + (\mathcal{R}v_{22})\sin^2\theta,$$

$$(\mathcal{P}^\star v) = -(\mathcal{R}v_{11})\sin\theta\cos\theta + (\mathcal{R}v_{12})(\cos^2\theta - \sin^2\theta) + (\mathcal{R}v_{22})\sin\theta\cos\theta.$$

Solve this system for $(\mathcal{R}v_{11})$, $(\mathcal{R}v_{12})$, and $(\mathcal{R}v_{22})$:

$$(\mathcal{R}v_{11}) = (\mathcal{P}v)\sin^2\theta - 2(\mathcal{P}^\star v)\sin\theta\cos\theta + (\mathcal{P}^\perp v)\cos^2\theta, \tag{11}$$

$$(\mathcal{R}v_{12}) = -(\mathcal{P}v)\sin\theta\cos\theta + (\mathcal{P}^\star v)(\cos^2\theta - \sin^2\theta) + (\mathcal{P}^\perp v)\sin\theta\cos\theta, \tag{12}$$

$$(\mathcal{R}v_{22}) = (\mathcal{P}v)\cos^2\theta + 2(\mathcal{P}^\star v)\sin\theta\cos\theta + (\mathcal{P}^\perp v)\sin^2\theta. \tag{13}$$

Prove (8); estimates (9) and (10) are established similarly. By Theorem 1,

$$(\mathcal{P}^\star((d^\perp)^2\psi)) = (\mathcal{P}^\perp((d^\perp)^2\psi)) = 0, \qquad (\mathcal{P}((d^\perp)^2\psi)) = (\mathcal{P}u).$$

Then from (11) we obtain $(\mathcal{R}((d^\perp)^2\psi)_{11}) = (\mathcal{P}u)\sin^2\theta$. Estimate $\|\mathcal{R}((d^\perp)^2\psi)_{11}\|^2_{H^1(Z)}$:

$$\|\mathcal{R}((d^\perp)^2\psi)_{11}\|^2_{H^1(Z)} = \int\limits_Z \left[(\mathcal{R}((d^\perp)^2\psi)_{11})^2 + \left(\frac{\partial}{\partial s}(\mathcal{R}((d^\perp)^2\psi)_{11})\right)^2 + \left(\frac{\partial}{\partial\theta}(\mathcal{R}((d^\perp)^2\psi)_{11})\right)^2\right] dsd\theta$$

$$= \int\limits_Z \left[(\mathcal{P}u)^2\sin^4\theta + \left(\frac{\partial(\mathcal{P}u)}{\partial s}\right)^2\sin^4\theta + \left(\frac{\partial(\mathcal{P}u)}{\partial\theta}\sin^2\theta + 2(\mathcal{P}u)\sin\theta\cos\theta\right)^2\right] dsd\theta.$$

Since $(a+b)^2 \le 2(a^2+b^2)$ for all a and b, we have

$$\|\mathcal{R}((d^\perp)^2\psi)_{11}\|^2_{H^1(Z)} \le \int\limits_Z (\mathcal{P}u)^2\sin^2\theta(1+7\cos^2\theta) + \sin^4\theta\left[\left(\frac{\partial(\mathcal{P}u)}{\partial s}\right)^2 + 2\left(\frac{\partial(\mathcal{P}u)}{\partial\theta}\right)^2\right] dsd\theta.$$

Taking it into account that $\sin^2\theta(1+7\cos^2\theta) \le 16/7$ and $\sin^4\theta \le 1$ for all θ, we obtain

$$\|\mathcal{R}((d^\perp)^2\psi)_{11}\|^2_{H^1(Z)} \le \frac{16}{7}\,\|\mathcal{P}u\|^2_{H^1(Z)}.$$

Analogously, it follows from (12) and (13) that

$$\|\mathcal{R}((d^\perp)^2\psi)_{12}\|^2_{H^1(Z)} \le 2\|\mathcal{P}u\|^2_{H^1(Z)}, \qquad \|\mathcal{R}((d^\perp)^2\psi)_{22}\|^2_{H^1(Z)} \le \frac{16}{7}\,\|\mathcal{P}u\|^2_{H^1(Z)}$$

respectively. By (6), we infer

$$\|((d^\perp)^2\psi)_{11}\|^2_{L_2(B)} \le C_1\|\mathcal{R}((d^\perp)^2\psi)_{11}\|^2_{H^1(Z)} \le \frac{16}{7}\,C_1\|\mathcal{P}u\|^2_{H^1(Z)},$$

$$\|((d^\perp)^2\psi)_{12}\|^2_{L_2(B)} \le C_1\|\mathcal{R}((d^\perp)^2\psi)_{12}\|^2_{H^1(Z)} \le 2C_1\|\mathcal{P}u\|^2_{H^1(Z)},$$

$$\|((d^\perp)^2\psi)_{22}\|^2_{L_2(B)} \le C_1\|\mathcal{R}((d^\perp)^2\psi)_{22}\|^2_{H^1(Z)} \le \frac{16}{7}\,C_1\|\mathcal{P}u\|^2_{H^1(Z)},$$

but

$$\|((d^\perp)^2\psi)_{11}\|^2_{L_2(B)} + 2\|((d^\perp)^2\psi)_{12}\|^2_{L_2(B)} + \|((d^\perp)^2\psi)_{22}\|^2_{L_2(B)} = \|(d^\perp)^2\psi\|^2_{L_2(S^2(B))},$$

and, therefore,

$$\|(d^\perp)^2\psi\|^2_{L_2(S^2(B))} \le \frac{60}{7}\,C_1\|\mathcal{P}u\|^2_{H^1(Z)} = C\|\mathcal{P}u\|^2_{H^1(Z)}.$$

Theorem 3 is proved. □

2.1. Formulas for Reconstructing the Potentials

The inversion formula for the Radon transform is well known:

$$f(x) = \frac{1}{4\pi^2}\int\limits_0^{2\pi}\int\limits_{-\infty}^{\infty} \frac{\partial^2(\mathcal{R}f)}{\partial s^2}(p - x^1\sin\theta + x^2\cos\theta, \theta)\ln|p|\,dpd\theta.$$

Since we consider $f \in H^k(B)$ in the present article, we have $(\mathcal{R}f)(s,\theta) = 0$ for $|s| > 1$. Consequently, the inversion formula can be rewritten as

$$f(x) = \frac{1}{4\pi^2}\int\limits_0^{2\pi}\int\limits_{P_{x,\theta}} \frac{\partial^2(\mathcal{R}f)}{\partial s^2}(p - x^1\sin\theta + x^2\cos\theta, \theta)\ln|p|\,dpd\theta, \qquad (14)$$

where $P_{x,\theta} = \{p \mid |p - x^1 \sin\theta + x^2 \cos\theta| \le 1\}$.

Theorem 4. *Let $u = (d^\perp)^2\psi + dd^\perp\phi + d^2\varphi$ with potentials $\varphi, \phi, \psi \in H_0^2(B)$. Then we have the formulas for reconstruction of φ, ϕ, and ψ:*

$$\psi(x) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_{P_{x,\theta}} [\mathcal{P}u]\big(p - x^1 \sin\theta + x^2 \cos\theta, \theta\big) \ln|p|\, dpd\theta,$$

$$\phi(x) = \frac{1}{2\pi^2} \int_0^{2\pi} \int_{P_{x,\theta}} [\mathcal{P}^\star u]\big(p - x^1 \sin\theta + x^2 \cos\theta, \theta\big) \ln|p|\, dpd\theta,$$

$$\varphi(x) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_{P_{x,\theta}} [\mathcal{P}^\perp u]\big(p - x^1 \sin\theta + x^2 \cos\theta, \theta\big) \ln|p|\, dpd\theta.$$

These formulas follow from (14) and Theorem 2.

2.2. Componentwise Inversion Formulas

If a ray with direction vector $\xi = (-\sin\theta, \cos\theta)$ and normal vector $\eta = (\cos\theta, \sin\theta)$ passes through the point $x = (x^1, x^2)$ then $s = x^1\cos\theta + x^2 \sin\theta$. Consequently, given a function f and a symmetric 2-tensor field u, we have

$$(\mathcal{R}f)(s,\theta) = (\mathcal{R}f)(x,\theta), \qquad (\mathcal{P}u)(s,\theta) = (\mathcal{P}u)(x,\theta),$$
$$(\mathcal{P}^\perp u)(s,\theta) = (\mathcal{P}^\perp u)(x,\theta), \qquad (\mathcal{P}^\star u)(s,\theta) = (\mathcal{P}^\star u)(x,\theta).$$

The inversion formula for $\mathcal{R}$ was obtained in [3]:

$$f(x) = \frac{1}{4\pi}(-\Delta)^{1/2} \int_0^{2\pi} (\mathcal{R}f)(x,\theta)\, d\theta, \tag{15}$$

where $(-\Delta)^{1/2}$ is the pseudodifferential operator for which

$$\mathcal{F}[(-\Delta)^{1/2} g](y) = |y|\mathcal{F}[g](y).$$

Henceforth, $\mathcal{F}[\cdot]$ stands for the application of the Fourier transform.

Let $u = (d^\perp)^2\psi + dd^\perp\phi + d^2\varphi$ with potentials $\varphi, \phi, \psi \in H_0^2(B)$. In [3], there were also obtained the following formulas for reconstructing the solenoidal part $(d^\perp)^2\psi$ of a symmetric 2-tensor field u:

$$((d^\perp)^2\psi)_{11}(x) = \frac{1}{8\pi}\left[(-\Delta)^{1/2} \int_0^{2\pi} (3\sin^2\theta - 1)(\mathcal{P}u)(x,\theta) d\theta - (-\Delta)^{-1/2} \int_0^{2\pi} \frac{\partial^2(\mathcal{P}u)}{\partial(x^1)^2}(x,\theta)\, d\theta\right],$$

$$((d^\perp)^2\psi)_{12}(x) = \frac{1}{8\pi}\left[(-\Delta)^{1/2} \int_0^{2\pi} (-3\sin\theta\cos\theta)(\mathcal{P}u)(x,\theta)\, d\theta - (-\Delta)^{-1/2} \int_0^{2\pi} \frac{\partial^2(\mathcal{P}u)}{\partial x^1 \partial x^2}(x,\theta)\, d\theta\right],$$

$$((d^\perp)^2\psi)_{22}(x) = \frac{1}{8\pi}\left[(-\Delta)^{1/2} \int_0^{2\pi} (3\cos^2\theta - 1)(\mathcal{P}u)(x,\theta)\, d\theta - (-\Delta)^{-1/2} \int_0^{2\pi} \frac{\partial^2(\mathcal{P}u)}{\partial(x^2)^2}(x,\theta)\, d\theta\right].$$

We significantly simplify these for $\mathcal{P}$ and obtain similar formulas for $\mathcal{P}^\perp$ and $\mathcal{P}^\star$.

Theorem 5. *Let $u = (d^\perp)^2\psi + dd^\perp\phi + d^2\varphi$ with potentials $\varphi, \phi, \psi \in H_0^2(B)$. Then the following holds for the reconstruction of the solenoidal part $(d^\perp)^2\psi$ of a symmetric 2-tensor field u:*

$$((d^\perp)^2\psi)_{ij}(x) = \frac{1}{4\pi}(-\Delta)^{1/2}\int_0^{2\pi} \xi^i\xi^j(\mathcal{P}u)(x,\theta)\,d\theta.$$

For the potential part $d^2\varphi$ of a symmetric 2-tensor field u, we have

$$(d^2\varphi)_{ij}(x) = \frac{1}{4\pi}(-\Delta)^{1/2}\int_0^{2\pi} \eta^i\eta^j(\mathcal{P}^\perp u)(x,\theta)\,d\theta.$$

For reconstructing the potential part $dd^\perp\phi$ of a symmetric 2-tensor field u, we have

$$(dd^\perp\phi)_{ij}(x) = \frac{1}{4\pi}(-\Delta)^{1/2}\int_0^{2\pi} (\xi^i\eta^j + \xi^j\eta^i)(\mathcal{P}^\star u)(x,\theta)\,d\theta.$$

Proof. By Theorem 1,

$$(\mathcal{P}^\star((d^\perp)^2\psi)) = (\mathcal{P}^\perp((d^\perp)^2\psi)) = 0, \qquad (\mathcal{P}((d^\perp)^2\psi)) = (\mathcal{P}u).$$

Then from (11)–(13) we obtain

$$(\mathcal{R}((d^\perp)^2\psi)_{11}) = (\mathcal{P}u)\sin^2\theta, \qquad (\mathcal{R}((d^\perp)^2\psi)_{12}) = (\mathcal{P}u)(-\sin\theta\cos\theta),$$
$$(\mathcal{R}((d^\perp)^2\psi)_{22}) = (\mathcal{P}u)\cos^2\theta,$$

i.e., $(\mathcal{R}((d^\perp)^2\psi)_{ij}) = \xi^i\xi^j(\mathcal{P}u)$. Applying (15), we infer

$$((d^\perp)^2\psi)_{ij}(x) = \frac{1}{4\pi}(-\Delta)^{1/2}\int_0^{2\pi}(\mathcal{R}((d^\perp)^2\psi)_{ij})(x,\theta)\,d\theta = \frac{1}{4\pi}(-\Delta)^{1/2}\int_0^{2\pi}\xi^i\xi^j(\mathcal{P}u)(x,\theta)\,d\theta.$$

The remaining two formulas are proved similarly. □

2.3. The Projection Theorem

Consider $(\mathcal{R}_\eta f)(s) = (\mathcal{R}f)(s,\theta)$ as a function of s for fixed θ. Recall that $\eta = (\cos\theta, \sin\theta)$. Formulate the so-called projection theorem for the Radon transform operator:

Proposition. *If $f \in C^\infty(B)$ then $\mathcal{F}[\mathcal{R}_\eta f](\sigma) = \sqrt{2\pi}\,\mathcal{F}[f](\sigma\eta)$, where $\sigma \in \mathbb{R}$.*

For the proof of the proposition, the reader is referred, for example, to [8].

Consider $(\mathcal{P}_\eta u)(s) = (\mathcal{P}u)(s,\theta)$, $\big(\mathcal{P}_\eta^\perp u\big)(s) = (\mathcal{P}^\perp u)(s,\theta)$ and $\big(\mathcal{P}_\eta^\star u\big)(s) = (\mathcal{P}^\star u)(s,\theta)$ as functions of s for some fixed θ.

Theorem 6. *If $\varphi \in H_0^2(B) \cap C^\infty(B)$ then*

$$\mathcal{F}[\mathcal{P}_\eta((d^\perp)^2\varphi)](\sigma) = -\sigma^2\sqrt{2\pi}\,\mathcal{F}[\varphi](\sigma\eta),$$
$$\mathcal{F}\big[\mathcal{P}_\eta^\perp(d^2\varphi)\big](\sigma) = -\sigma^2\sqrt{2\pi}\,\mathcal{F}[\varphi](\sigma\eta),$$
$$\mathcal{F}\big[\mathcal{P}_\eta^\star(dd^\perp\varphi)\big](\sigma) = -\sigma^2\sqrt{\pi/2}\,\mathcal{F}[\varphi](\sigma\eta),$$

where $\sigma \in \mathbb{R}$.

Proof. By the proposition,

$$\mathcal{F}[\mathcal{R}_\eta\varphi](\sigma) = \sqrt{2\pi}\,\mathcal{F}[\varphi](\sigma\eta), \qquad \sigma \in \mathbb{R}.$$

Theorem 2 and the properties of the Fourier transform

$$\mathcal{F}\left[\frac{\partial^2 f}{\partial t^2}\right](w) = (iw)^2\mathcal{F}[f](w)$$

yield

$$\mathcal{F}[\mathcal{P}_\eta((d^\perp)^2\varphi)](\sigma) = \mathcal{F}\left[\frac{\partial^2(\mathcal{R}_\eta\varphi)}{\partial s^2}\right](\sigma) = (i\sigma)^2\mathcal{F}[\mathcal{R}_\eta\varphi](\sigma) = -\sigma^2\sqrt{2\pi}\mathcal{F}[\varphi](\sigma\eta), \qquad \sigma \in \mathbb{R}.$$

The remaining two formulas are proved similarly. □

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